TA NOTES

38. **Euler Equation.** An equation of the form

$$t^2y'' + \alpha ty' + \beta y = 0,\ t > 0,$$

where α and β are real constants, is called an Euler equation. Show that the substitution $x = \ln t$ transform an Euler equation into an equation with constant coefficients.

Answer:

$$\frac{dx}{dt} = \frac{1}{t}$$

$$\frac{d^2y}{dt^2} = \frac{1}{t^2}\frac{d^2y}{dx^2} - \frac{1}{t^2}\frac{dy}{dx}$$

We get

$$t^2y'' + \alpha ty' + \beta y = \frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0.$$

❑

In each of Problems, using the result of problem 38 to solve the given equation for $t > 0$.

40. $t^2y'' + 4ty' + 2y = 0$

42. $t^2y'' - 4ty' - 6y = 0$

Answer: 40. Let $x = \ln t$, by Problem 38, we have

$$\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 0 \tag{1}$$

The characteristic equation of (1) is

$$r^2 + 5r + 2 = 0$$

Thus the possible values of r are $r_1 = -\frac{5}{2} + \frac{\sqrt{17}}{2}$ and $r_2 = -\frac{5}{2} - \frac{\sqrt{17}}{2}$, and the general solution of the equation (1) is

$$y(x) = c_1e^{(-\frac{5}{2}+\frac{\sqrt{17}}{2})x} + c_2e^{(-\frac{5}{2}-\frac{\sqrt{17}}{2})x}.$$

Then the solution of original equation is

$$\begin{aligned} y(t) = &\quad c_1 e^{(-\frac{5}{2}+\frac{\sqrt{17}}{2})\ln t} + c_2 e^{(-\frac{5}{2}-\frac{\sqrt{17}}{2})\ln t} \\ = &\quad c_1 t^{-\frac{5}{2}+\frac{\sqrt{17}}{2}} + c_2 t^{-\frac{5}{2}-\frac{\sqrt{17}}{2}} \end{aligned}$$

42. Let $x = \ln t$, by Problem 38, we have

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 6y = 0 \tag{1}$$

The characteristic equation of (1) is

$$r^2 - 3r - 6 = 0$$

Thus the possible values of r are $r_1 = \frac{3}{2} + \frac{\sqrt{33}}{2}$ and $r_2 = \frac{3}{2} - \frac{\sqrt{33}}{2}$, and the general solution of the equation (1) is

$$y(x) = c_1 e^{(\frac{3}{2}+\frac{\sqrt{33}}{2})x} + c_2 e^{(\frac{3}{2}-\frac{\sqrt{33}}{2})x}.$$

Then the solution of original equation is

$$\begin{aligned} y(t) = &\quad c_1 e^{(\frac{3}{2}+\frac{\sqrt{33}}{2})\ln t} + c_2 e^{(\frac{3}{2}-\frac{\sqrt{33}}{2})\ln t} \\ = &\quad c_1 t^{\frac{3}{2}+\frac{\sqrt{33}}{2}} + c_2 t^{\frac{3}{2}-\frac{\sqrt{33}}{2}} \end{aligned}$$

❏

33. The method of problem 20 can be extend to second order equation with variable coefficients. If y_1 is a known nonvanishing solution of $y'' + p(t)y' + q(t)y = 0$, show that a second solution y_2 satisfies $(y_2/y_1)' = W(y_1, y_2)/y_1^2$, where $W(y_1, y_2)$ is the Wronskian of y_1 and y_2. Then use Abel's formula [Eq. (8) of section 3.3] to determine y_2.

Answer: By direct computation, we have

$$(\frac{y_2}{y_1})' = \frac{W(y_1, y_2)}{y_1^2}.$$

By Abel's formula, $W' = -pW$, hence $W = c_1 e^{-\int^x p(s)ds}$. Then

$$\begin{aligned} \frac{y_2}{y_1} = &\quad \int^x \frac{c_1 e^{-\int^t p(s)ds}}{y_1^2(t)} dt + c_2 \\ \Rightarrow\ y_2(x) = &\quad c_1 y_1(x) \int^x \frac{e^{-\int^t p(s)ds}}{y_1^2(t)} dt + c_2 y_1(x) \end{aligned}$$

❏

In the following problem use the method of of problem 33 to find second independent solution of the given equation.

35. $ty'' - y' + 4t^3y = 0;\ t > 0;\ y_1(t) = \sin(t^2)$

36. $(x-1)y'' - xy' + y = 0,\ x > 1,\ y_1(x) = e^x$

Answer: 35. The original equation can be written as

$$y'' - \frac{}{t}y' + 4t^2y = 0.$$

From the Abel's theorem

$$W(y_1, y_2) = c_1 e^{\int \frac{dt}{t}} = c_1 t$$

We get

$$(\frac{y_2}{y_1})' = \frac{W(y_1, y_2)}{y_1^2} = \frac{c_1 t}{\sin^2 t^2}$$

and

$$y_2(t) = c_1 \sin(t^2) \int \frac{t}{\sin^2 t^2} dt + c_2 \sin(t^2).$$

36. The original equation can be written as

$$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0.$$

Here $p(x) = \frac{x}{x-1}$, $q(x) = \frac{1}{x-1}$. By the answer of problem, we get

$$\begin{aligned} y_2(x) = & \quad c_1 y_1(x) \int^x \frac{e^{-\int^t p(s)ds}}{y_1^2(t)} dt + c_2 y_1(x) \\ = & \quad c_1 e^x \int^x \frac{e^{-\int^t \frac{s}{s-1} ds}}{e^{2t}} dt + c_2 e^x \\ = & \quad c_1 e^x \int^x (t-1)e^{-t} dt + c_2 e^x \\ = & \quad c_1 x + c_2 e^x \end{aligned}$$

❑

28. Determined the general solution of

$$y'' + \lambda^2 y = \sum_{m=1}^{N} a_m \sin m\pi t, \qquad \text{(i)}$$

where $\lambda > 0$ and $\lambda \neq m\pi$ for $m = 1, \cdots, N$.

Answer: It is easy to know that the general solution to the corresponding homogeneous equation is $c_1 \cos \lambda t + c_2 \sin \lambda t$. If we find a particular solution, then we are done.

Note that, if $y_m(t)$ satisfy $y_m'' + \lambda^2 y_m = a_m \sin m\pi t$, then, $y(t) = \sum_{m=1}^{N} y_m(t)$ is a particular solution to the Eq. (i). Therefore, we reduce the problem to find a particular solution to

$$y_m'' + \lambda^2 y_m = a_m \sin m\pi t \tag{ii}$$

which is much easier. Clearly, we can assume that $y_m(t) = c_m \cos m\pi t + d_m \sin m\pi t$. Then

$$(\lambda^2 - m^2\pi^2)c_m \cos m\pi t + (\lambda^2 - m^2\pi^2)d_m \sin m\pi t = a_m \sin m\pi t.$$

Therefore, we can take

$$c_m = 0, \quad d_m = \frac{a_m}{\lambda^2 - m^2\pi^2}.$$

Then $y_m(t) = \frac{a_m}{\lambda^2 - m^2\pi^2} \sin m\pi t$ is a particular solution to the Eq. (ii). Hence

$$y(t) = \sum_{m=1}^{N} y_m(t) = \sum_{m=1}^{N} \frac{a_m}{\lambda^2 - m^2\pi^2} \sin m\pi t$$

is a particular solution to Eq. (i). Therefore, the general solution to the Eq. (i) is

$$y(t) = c_1 \cos \lambda t + c_2 \sin \lambda t + \sum_{m=1}^{N} \frac{a_m}{\lambda^2 - m^2\pi^2} \sin m\pi t.$$

❑

30. Follow the instructions in Problem 29 to solve the differential equation

$$y'' + 2y' + 5y = \begin{cases} 1, & 0 \le t \le \pi/2, \\ 0, & t > \pi/2 \end{cases}$$

with the initial conditions $y(0) = 0$ and $y'(0) = 0$.

Answer: The idea is that: Solve the equation in $[0, \pi/2]$ with the initial condition $y(0) = 0$ and $y'(0) = 0$ first. Then, solve the equation for $t > \pi/2$ with the initial condition $y(\pi/2), y'(\pi/2)$.

The general solution of the corresponding ODE is $c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t$. For $0 < t < \pi/2$, the ODE is

$$y'' + 2y' + 5y = 1.$$

A particular solution is $Y = 1/5$. Therefore, in $[0, \pi/2]$, the general solution is

$$y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t + \frac{1}{5}, \quad \text{for } 0 \le t \le \frac{\pi}{2}.$$

Apply the initial condition, one can determined the $c_1 = -1/5, c_2 = -1/10$. That is in $[0, \pi/2]$, the solution is

$$y(t) = -\frac{1}{5} e^{-t} \cos 2t - \frac{1}{10} e^{-t} \sin 2t + \frac{1}{5}, \quad \text{for } 0 \le t \le \frac{\pi}{2}$$

It gives

$$y(\frac{\pi}{2}) = 1/5(1 + e^{-\pi/2}), \quad y'(\frac{\pi}{2}) = 0. \tag{i}$$

For $t > \pi/2$, the ODE reads

$$y'' + 2y' + 5y = 0$$

The general solution is

$$y(t) = c_1 e^{-t} \cos 2t + c_2 e^{-t} \sin 2t, \quad \text{for } t \ge \frac{\pi}{2}.$$

Apply the initial condition (i), one has

$$c_1 = -\frac{1}{5}(1 + e^{\frac{\pi}{2}}), \quad c_2 = -\frac{1}{10}(1 + e^{\frac{\pi}{2}}).$$

Then, the solution for $t > \pi/2$ is

$$y(t) = -\frac{1}{5}(1 + e^{\frac{\pi}{2}}) e^{-t} \cos 2t - \frac{1}{10}(1 + e^{\frac{\pi}{2}}) e^{-t} \sin 2t, \quad \text{for } t \ge \frac{\pi}{2}.$$

Therefore the solutio to the original problem is

$$y(t) = \begin{cases} -\frac{1}{5} e^{-t} \cos 2t - \frac{1}{10} e^{-t} \sin 2t + \frac{1}{5}, & \text{for } 0 \le t \le \frac{\pi}{2} \\ -\frac{1}{5}(1 + e^{\frac{\pi}{2}}) e^{-t} \cos 2t - \frac{1}{10}(1 + e^{\frac{\pi}{2}}) e^{-t} \sin 2t, & \text{for } t \ge \frac{\pi}{2}. \end{cases}$$

❑

38. If a, b, c are positive constants, show that all solutions of $ay'' + by' + cy = 0$ approach zero as $t \to \infty$.

Answer: The characteristic equation is

$$ar^2 - br + c = 0$$

Case A: $b^2 - 4ac < 0$

Thus the possible values of r are $r_1 = \frac{-b+i\sqrt{4ac-b^2}}{2a}$ and $r_2 = \frac{-b-i\sqrt{4ac-b^2}}{2a}$, and the general solution of the equation is

$$y(t) = e^{\frac{-bt}{2a}}(c_1 \cos \frac{\sqrt{4ac-b^2}}{2a} t + c_2 \sin \frac{\sqrt{4ac-b^2}}{2a} t).$$

Hence, $y(t) \to 0,$ as $t \to \infty$, since $a,\ b$ are positive constants.

Case B: $b^2 - 4ac = 0$

Thus the possible values of r are $r_1 = \frac{-b}{2a}$, and the general solution of the equation is

$$y(t) = e^{\frac{-bt}{2a}}(c_1 + c_2 t).$$

Hence, $y(t) \to 0,$ as $t \to \infty$, since $a,\ b$ are positive constants.

Case C: $b^2 - 4ac > 0$

Thus the possible values of r are $r_1 = \frac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2 = \frac{-b-\sqrt{b^2-4ac}}{2a}$, and the general solution of the equation is

$$y(t) = c_1 e^{\frac{-b+\sqrt{b^2-4ac}}{2a}} + c_2 e^{\frac{-b-\sqrt{b^2-4ac}}{2a}}.$$

Hence, $y(t) \to 0,$ as $t \to \infty$, since a is positive constant and $-b + \sqrt{b^2 - 4ac} < 0$. ❏

Consider the differential equation

$$ay'' + by' + cy = g(t) \tag{i}$$

where a, b and c are positive.

31. If $Y_1(t)$ and $Y_2(t)$ are solutions of Eq. (i), show that $Y_1(t) - Y_2(t) \to 0$ as $t \to \infty$. Is this result true if $b = 0$?

Answer: Let $u = Y_1(t) - Y_2(t)$, clearly, u is the solution of the homogeneous equation

$$ay'' + by' + cy = 0.$$

Hence by the problem 38 of section 3.5, we know that

$$Y_1(t) - Y_2(t) = u(t) \to 0.$$

By (a) part of problem 39 of section 3.5, we know that the result is false for $b = 0$. ❏

vskip 1cm In each of the problems verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

14. $t^2y'' - t(t+2)y' + (t+2)y = 2t^3,\ t > 0;\ y_1 = t,\ y_2 = te^t$

16. $(1-t)y'' + ty' - y = 2(t-1)^2e^{-t},\ 0 < t < 1;\ y_1 = e^t,\ y_2 = t$

19. $(1-x)y'' + xy' - y = g(x),\ 0 < x < 1;\ y_1 = e^x,\ y_2 = x$

Answer: 14. It is easy to check that y_1 and y_2 satisfy

$$t^2y'' - t(t+2)y' + (t+2)y = 0.$$

$W(y_1, y_2) = t^2e^t$ and let

$$\begin{aligned} Y(t) &= -t\int \frac{se^s2s}{s^2e^s}ds + te^t\int \frac{s2s}{s^2e^s}ds \\ &= -2t^2 - 2t. \end{aligned}$$

then a particular solution of the original equation is

$$y(t) = -2t^2 - 2t.$$

16. It is easy to check that y_1 and y_2 satisfy

$$(1-t)y'' + ty' - y = 0$$

$W(y_1, y_2) = e^t(1-t)$ and let

$$\begin{aligned} Y(t) &= -e^t\int \frac{s2(1-s)e^{-s}}{(1-s)e^s}ds + t\int \frac{e^s2(1-s)e^{-s}}{(1-s)e^s}ds \\ &= -te^{-t} + \frac{1}{2}e^{-t}. \end{aligned}$$

then a particular solution of the original equation is

$$y(t) = -te^{-t} + \frac{1}{2}e^{-t}.$$

19. It is easy to check that y_1 and y_2 satisfy

$$(1-x)y'' + xy' - y = 0$$

$W(y_1, y_2) = (1-x)e^x$ and let

$$Y(t) = -e^x \int \frac{g(s)s}{(1-s)^2 e^s} ds + x \int \frac{g(s)e^s}{(1-s)^2 e^s} ds$$
$$= -e^x \int \frac{g(s)s}{(1-s)^2 e^s} ds + x \int \frac{g(s)}{(1-s)^2} ds.$$

then a particular solution of the original equation is

$$y(t) = -e^x \int \frac{g(s)s}{(1-s)^2 e^s} ds + x \int \frac{g(s)}{(1-s)^2} ds.$$

❏